Gaussian Integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \tag{1}$$

$$\int_{0}^{\infty} e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{2}$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}} \tag{3}$$

$$\int_{0}^{\infty} e^{iax^2}\,dx = \frac{1}{2}\sqrt{\frac{i\pi}{a}} \tag{4}$$

$$\int_{0}^{\infty} e^{-iax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{ia}} \tag{5}$$

In general, from dimensional anlysis we see:

$$\int_{0}^{\infty} x^n e^{-ax^2}\,dx \quad \propto \quad a^{-(\frac{n+1}{2})} \tag{6}$$

and in particular:

$$\int_{0}^{\infty} x^n e^{-ax^2}\,dx = \begin{cases} \frac{(n-1)\cdot(n-3)\ldots 3\cdot 1}{2^{\frac{n}{2}+1}a^{\frac{n}{2}}}\sqrt{\frac{\pi}{a}}, & \text{for n even} \\ \frac{[\frac{1}{2}(n-1)]!}{2a^{\frac{n+1}{2}}}, & \text{for n odd} \end{cases} \tag{7}$$

Notes on proving these integrals: Integral 1 is done by squaring the integral, combining the exponents to $x^2 + y^2$ switching to polar coordinates, and taking the R integral in the limit as $R \to \infty$. Integral 2 is done by changing variables then using Integral 1. Integral 3 is done by completing the square in the exponent and then changing variables to use equation 1. Integral 4(5) can be done by integrating over a wedge with angle $\frac{\pi}{4}(-\frac{\pi}{4})$, using Cauchy's theory to relate the integral over the real number to the other side of the wedge, and then using Integral 1.

For n even Integral 7 can be done by taking derivatives of equation 2 with respect to a. For n odd, Integral 7 can be done with the substitution $u = ax^2$, and then integrating by parts.