A Derivation of the Quantum Mechanical Momentum Operator in the Position Representation

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1 Translation Operator

Given an eigenstate of position $|\vec{x}\rangle$, with eigenvalue x, we define a *Translation Operator*, $T(\vec{a})$, which transforms an eigenstate of position to another eigenstate of position, with the eigenvalue increased by $\vec{a}$.

$$T(\vec{a})\,|\vec{x}\rangle \equiv |\vec{x}+\vec{a}\rangle \tag{1}$$

By the following argument, we note that the adjoint of $T(\vec{a})$ moves a state backward. It transforms an eigenstate of position to another eigenstate of position, with the eigenvalue decreased by $\vec{a}$.

$$\begin{aligned}
\langle \vec{x}' \,|\, T(\vec{a}) \,|\, \vec{x}\rangle &= \langle \vec{x}' \,|\, \vec{x}+\vec{a}\rangle && (2)\\
&= \delta((\vec{x}+\vec{a})-\vec{x}') && (3)\\
&= \delta(\vec{x}-(\vec{x}'-\vec{a})) && (4)\\
&= \langle \vec{x}'-\vec{a} \,|\, \vec{x}\rangle && (5)
\end{aligned}$$

$$\Rightarrow \quad \langle \vec{x}' |\, T(\vec{a}) = \langle \vec{x}'-\vec{a}| \tag{6}$$

$$T^{\dagger}(\vec{a})\,|\vec{x}'\rangle = |\vec{x}'-\vec{a}\rangle \tag{7}$$

Note that if we translate forwards by some amount, it is the same as translating backwards by negative that amount.

$$T(\vec{a}) = T^{\dagger}(-\vec{a}) \tag{8}$$

If we translate a state forwards and then backwards by the same amount, the state remains unchanged. This implies that the translation operator is unitary.

$$T^\dagger(\vec{a})\, T(\vec{a})\, |\vec{x}\rangle = |\vec{x}\rangle \tag{9}$$

$$\Rightarrow \quad T^\dagger(\vec{a}) = T^{-1}(\vec{a}) \tag{10}$$

Any unitary operator can be written as

$$T(\vec{a}) = e^{-i\vec{K}\cdot\vec{a}} \tag{11}$$

$$1 = T^\dagger(\vec{a})\, T(\vec{a}) \tag{12}$$

$$= e^{i\vec{K}^\dagger\cdot\vec{a}}\, e^{-i\vec{K}\cdot\vec{a}} \tag{13}$$

$$= e^{i(\vec{K}^\dagger-\vec{K})\cdot\vec{a}} \tag{14}$$

$$\Rightarrow \quad \vec{K} = \vec{K}^\dagger \tag{15}$$

Where evidently, $\vec{K}$ must be hermitian. In general, when writing a unitary operator this way, the operators $\vec{K}$ are known as the *generators* of what ever unitary operator one is expressing, in this case: translation.

2 Eigenstates of $\vec{K}$

Let us call the eigenstates of $\vec{K}$, which are also eigenstates of $T(\vec{a})$, $|\vec{k}\rangle$.

$$\vec{K}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle \quad \text{and} \quad T(\vec{a})\,|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}\,|\vec{k}\rangle \tag{16}$$

Let us consider the position projection of the translation operator acting on an eigenstate of translation. Letting the translation operator, operate to the right, we have

$$\langle\vec{x}|T(\vec{a})|\vec{k}\rangle = e^{-i\vec{k}\cdot\vec{a}}\ \langle\vec{x}|\vec{k}\rangle \tag{17}$$

$$= e^{-i\vec{k}\cdot\vec{a}}\ \psi_{\vec{k}}(\vec{x}) \tag{18}$$

where we have defined the *wavefunction* to be

$$\psi_{\vec{k}}(\vec{x}) = \langle\vec{x}|\vec{k}\rangle \tag{19}$$

Now consider the same projection, replacing $T(\vec{a})$ with $T^\dagger(-\vec{a})$, and letting it operate to the left.

$$\begin{aligned}\langle\vec{x}|T(\vec{a})|\vec{k}\rangle &= \langle\vec{x}|T^\dagger(-\vec{a})|\vec{k}\rangle && (20)\\ &= \langle\vec{x}-\vec{a}|\vec{k}\rangle && (21)\\ &= \psi_{\vec{k}}(\vec{x}-\vec{a}) && (22)\end{aligned}$$

Equating the two methods, we have

$$\psi_{\vec{k}}(\vec{x}-\vec{a}) = e^{-i\vec{k}\cdot\vec{a}}\ \ \psi_{\vec{k}}(\vec{x}) \tag{23}$$

Letting $\vec{x}=0$, and $\vec{a}=-\vec{y}$, we recognize that this gives plane wave solutions for the wavefunction.

$$\psi_{\vec{k}}(\vec{y}) = \psi_{\vec{k}}(0)\ e^{i\vec{k}\cdot\vec{y}} \tag{24}$$

As hypothesized by de Broglie, and first experimentally verified by electron diffraction, a particle in an eigenstate of momentum has a wavefunction with with a wavevector, $\vec{k}$, related to its momentum $\vec{p}$ by

$$\vec{p} = \hbar\,\vec{k} \tag{25}$$

This means that the $\vec{K}$ operator that we have been discussing is indeed the wavevector operator. We can now write the translation operator as

$$T(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}/\hbar} \tag{26}$$

Aside from the constant, $\hbar$, *momentum is the generator of translation.*

3 Matrix Elements of $\vec{P}$ in the $|\vec{x}\rangle$ Basis

For simplicity, let us now consider translation in only one dimension.

$$T(a) = e^{-iPa/\hbar} \tag{27}$$

The following clever manipulation reveals how to write the momentum operator in terms of the translation operator.

$$\left.\frac{\partial}{\partial a}\right|_{a=0} T(a) = -\frac{i}{\hbar}P \tag{28}$$

$$P = i\hbar \left.\frac{\partial}{\partial a}\right|_{a=0} T(a) \tag{29}$$

We should now ask what the matrix elements are of the momentum operator in the position basis.

$$\begin{aligned}
\langle x' \,|\, P \,|\, x \rangle &= i\hbar \left.\frac{\partial}{\partial a}\right|_{a=0} \langle x' \,|\, T(a) \,|\, x \rangle & (30) \\
&= i\hbar \left.\frac{\partial}{\partial a}\right|_{a=0} \delta(x + a - x') & (31) \\
&= i\hbar\, \delta'(x - x') & (32)
\end{aligned}$$

4 $\vec{P}$ Acting on a Wavefunction

We should now take a digression to investigate what is meaning of this derivative of a delta function, $\delta'(x)$. We integrate by parts, a $\delta'(x-y)$ acting on some arbitrary function, $f(x)$. Note that the boundary term is zero because $\delta(x-y)$ is zero on the boundary, provided a boundary of integration is not at position y.

$$\begin{aligned}
\int \delta'(x-y)\, f(x)\, dx &= 0 - \int \delta(x-y)\, f'(x)\, dx & (33) \\
&= -f'(y) & (34)
\end{aligned}$$

Evidently, the derivative of a delta function is sort of a tool for evaluating the derivative of some function at a certain point.

Now we may ask how we can represent the momentum operator in the position basis. Because the number of states in the position basis are uncountably infinite, a matrix representation would be awkward. We see by the following argument that there is a much more elegant way of writing the momentum operator.

Consider the momentum operator acting on the wavefunction of some state state $|\psi\rangle$.

$$\begin{aligned}
P\,\psi(x) &= \langle x \,|\, P \,|\, \psi\rangle && (35)\\
&= \int \langle x|P|x'\rangle\, \langle x'|\psi\rangle\, dx' && (36)\\
&= i\hbar \int \delta'(x'-x)\, \psi(x')\, dx' && (37)\\
&= -i\hbar\, \left.\frac{\partial \psi(x')}{\partial x'}\right|_{x'=x} && (38)\\
&= -i\hbar\, \frac{\partial \psi(x)}{\partial x} && (39)
\end{aligned}$$

$$\therefore \quad P \to -i\hbar\, \frac{\partial}{\partial x} \qquad (40)$$