Chapter 1

Preliminaries

1.1 Overview of Special Relativity

1.1.1 Lorentz Boosts

Searches in the later part 19th century for the coordinate transformation that left the form of Maxwell’s equations and the wave equation invariant lead to the discovery of the Lorentz Transformations. The “boost” transformation from one (unprimed) inertial frame to another (primed) inertial frame moving with dimensionless velocity $\vec{\beta} = \vec{\beta}/c$, respect to the former frame, is given by

$$(c t', x') = \left( \begin{array}{cc} \gamma & -\gamma \beta \\ -\gamma \beta & \gamma \end{array} \right) \left( \begin{array}{c} c t \\ x \end{array} \right)$$

Because a boost along one of the spacial dimensions leaves the other two unchanged, we can suppress the those two spacial dimensions and let $\beta = |\beta|$. $\gamma$ is the Lorentz Factor, defined by

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \quad (1.1)$$

$\gamma$ ranges from 1 to $\infty$ monotonically in the nonrelativistic ($\beta \to 0$) and relativistic ($\beta \to 1$) limits, respectively. It is useful to remember that $\gamma \geq 1$. Note that being the magnitude of a vector, $\beta$ has a lower limit at 0. $\beta$ also has an upper limit at 1 because $\gamma$ diverges as $\beta$ approaches 1 and becomes unphysically imaginary for values of $\beta > 1$. This immediately reveals that $\beta = 1$, or $v = c$, is Nature’s natural speed limit.
The inverse transformation is given by
\[
\begin{pmatrix}
  c t \\
  x
\end{pmatrix} = \begin{pmatrix}
  \gamma & \gamma \beta \\
  \gamma \beta & \gamma
\end{pmatrix} \begin{pmatrix}
  c t' \\
  x'
\end{pmatrix}
\]

1.1.2 Length Contraction and Time Dilation

The differences between two points in spacetime follow from the transformations:

\[
\begin{align*}
  c \Delta t' &= \gamma c \Delta t - \gamma \beta \Delta x \\
  \Delta x' &= -\gamma \beta c \Delta t + \gamma \Delta x \\
  c \Delta t &= \gamma c \Delta t' + \gamma \beta \Delta x' \\
  \Delta x &= \gamma \beta c \Delta t' + \gamma \Delta x'
\end{align*}
\]

Consider a clock sitting at rest in the unprimed frame ($\Delta x = 0$). The first of the four above equations and the fact that $\gamma \geq 1$, imply that the time interval is dilated in the primed frame.

\[
\Delta t' = \gamma \Delta t
\]

Now consider a rod of length $\Delta x$ in the unprimed frame. A measurement of the length in the primmed frame corresponds to determining the coordinates of the endpoints simultaneously in the unprimed frame ($\Delta t' = 0$). Then the fourth equation implies that length is contracted in the primed frame.

\[
\Delta x' = \frac{\Delta x}{\gamma}
\]

We call time intervals and lengths “proper” if they are measured in the frame where the subject is at rest (in this case, the unprimed frame). In summary, proper times and lengths are the shortest and longest possible, respectively.

1.1.3 Four-vectors

Knowing that lengths and times transform from one reference frame to another, one can wonder if there is anything that is invariant. Consider the
following, using the last two of the four equations for the differences between two spacetime points.

\[
(c \Delta t)^2 - (\Delta x)^2 = (\gamma c \Delta t' + \gamma \beta \Delta x')^2 - (\gamma \beta c \Delta t' + \gamma \Delta x')^2 \\
= \gamma^2 \left[(c \Delta t')^2 + 2\beta c \Delta t' \Delta \bar{x}' + \beta^2 (\Delta x')^2 - \beta^2 (c \Delta t')^2 - 2\beta \Delta t' \Delta \bar{x}' - (\Delta x')^2\right] \\
= \gamma^2 \left[(c \Delta t')^2 - (\Delta x')^2\right] \\
= (c \Delta t')^2 - (\Delta x')^2 \equiv (\Delta \tau)^2
\]

Which shows that \(\Delta \tau\) has the same value in any frames related by Lorentz Transformations. \(\Delta \tau\) is called the "\textbf{invariant length.}" Note that it is equal to the proper time interval.

This motivates us to think of \((t, \bar{x})\) as a \textbf{four-vector} that transforms according to the Lorentz transformations, in a "spacetime vector space," and there should be some kind of "inner product," or contraction, of these vectors that leaves \(\Delta \tau\) a scalar. This can be done by defining the Minkowski metric tensor as follows.

\[
\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \tag{1.2}
\]

Four-vectors are indexed by a Greek index, \(x^\mu = (c t, \bar{x})^\mu, \mu \text{ ranging from 0 to 3 (} x^0 = c t, x^1 = x, x^2 = y, x^3 = z).\) The contraction of a spacetime four-vector with itself, its square, is give by

\[
x^\mu x_\mu \equiv x^\mu \eta_{\mu\nu} x^\nu = (c t)^2 - \bar{x} \cdot \bar{x} = (\Delta \tau)^2 \tag{1.3}
\]

giving the square of the invariant length between \(x^\mu\) and the origin. In equation (1.3), we have defined that the lowering of a four-vector index is done by multiplication by the metric tensor. Explicit matrix multiplication will show that \(\eta^{\mu\nu}\) is the inverse Minkowski metric and has the same components as \(\eta_{\mu\nu}.\)

\[
\eta_{\mu\lambda} \eta^{\lambda\nu} = \delta^\nu_\mu \tag{1.4}
\]

Raising the indices of the metric confirms that the components of the metric and inverse metric are equal.

\[
\eta^{\mu\nu} = \eta^{\mu\lambda} \eta_{\nu\sigma} \eta_{\lambda\sigma} = \eta^{\mu\lambda} \eta_{\lambda\sigma} (\eta^T)^{\sigma\nu}
\]
Anything that transforms according to the Lorentz Transformations, like $(c t, \vec{x})$, is a four-vector. Another example of a four-vector is four-velocity, defined by

$$u^\mu \equiv \gamma (c, \vec{v})^\mu \tag{1.5}$$

One can show that the square of $u^\mu$ is invariant as required.

$$u^\mu u_\mu = \gamma^2 (c^2 - v^2)$$
$$= \frac{1}{1 - \beta^2} (1 - \beta^2) c^2$$
$$= c^2$$

which is obviously invariant. Any equation where all of the factors are scalars (with no indices or contracting indices), or are four-vectors/tensors, with matching indices on the other side of the equal sign, is called “manifestly invariant.”

### 1.1.4 Momentum and Energy

The Classically conserved definitions of momentum and energy, being dependent on the coordinate frame, will not be conserved in other frames. We are motivated to consider the effect of defining momentum with the four-velocity instead of the classical velocity. The mass of a particle, $m$, being an intrinsic property of the particle, must be a Lorentz scalar. Therefore, the following definition of the four-momentum is manifestly a four-vector.

$$p^\mu \equiv m u^\mu = \gamma m (c, \vec{v})^\mu \tag{1.6}$$

The square of which is

$$p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 \tag{1.7}$$

Now let’s give some interpretation to the components of the four-momentum. To consider the nonrelativistic limit, let us expand $\gamma$ in the $\beta \to 0$ limit.

$$\gamma \simeq 1 + \frac{1}{2} \beta^2 + \frac{3}{8} \beta^4 + \cdots$$

Then the leading order term of the space-like components of the four-momentum is just the Classical momentum.

$$\vec{p} = m \vec{v} + \cdots$$
We therefore, interpret the space-like components of the four-momentum as the relativistic momentum.

\[ \vec{p} = \gamma m \vec{v} \]  

(1.8)

The expansion of the time-like term gives

\[ m c^2 \frac{1}{2} m v^2 + \cdots \]

We can now recognize the second term as the Classical kinetic energy. The first term is evidently the “mass energy,” energy present even when \( v = 0 \). Higher order terms give relativistic corrections.

\[ E = \gamma m c^2 \]  

(1.9)

We can therefore write the four-momentum in terms of the relativistic energy, \( E \), and relativistic momentum, \( p \).

\[ p^\mu = (E, \vec{p})^\mu \]  

(1.10)

The four-momentum is the combination of momentum and energy necessary to transform according to Lorentz Transformations. Both \( E \) and \( \vec{p} \) are conserved quantities in any given frame, but they are not invariant; they transform when going to another frame. Scalar quantities, like mass, are invariant but are not necessarily conserved. Mass can be exchanged for kinetic energy and vice versa. Charge is an example of a scalar quantity that is also conserved.

Looking at the square of the four-momentum with this energy-momentum interpretation of its components gives the very important relationship between energy, momentum, and mass.

\[ p^\mu p_\mu = E^2 - |\vec{p}|^2 c^2 = m^2 c^4 \]  

(1.11)

Taking the ratio of equations (1.8) and (1.9) gives the following interesting relation.

\[ \frac{\vec{p}}{E} = \frac{\vec{v}}{c^2} \]  

(1.12)

which leads to

\[ \frac{\vec{p} c}{E} = \vec{\beta} \]  

(1.13)
Note from equation (1.11), in the case that \( m = 0 \), we have that \( E = |\vec{p}|c \), which implies that \( \beta = 1 \). Therefore, massless particles must travel at the speed of light. In which case (1.13) agrees that the following is the energy-momentum relation for massless particles.

\[
E = |\vec{p}|c
\]  
(1.14)

Also note the relationship to Einstein’s equation for the energy of a photon, \( E = h \omega \Rightarrow |\vec{p}| = \frac{h \omega}{c} = \hbar k \), consistent with de Broglie’s relation.

1.2 Units

1.2.1 Natural Units

Factors of \( c \) were explicit in the above review of special relativity. From now on, we will use a form of natural units, where certain natural constants are set to one by using units derived from the God-given scales in Nature.

\[
\hbar = c = \varepsilon_0 = 1
\]  
(1.15)

From \( \hbar = 6.58 \times 10^{-25} \text{ GeV} \cdot \text{s} = 1 \), it follows that if we choose to measure energy in units GeV, then time can be measured in units GeV\(^{-1}\).

\[
1 \text{ GeV}^{-1} = 6.58 \times 10^{-25} \text{ s}
\]  
(1.16)

From \( c = 3 \times 10^8 \text{ m/s} = 1 \), it follows that

\[
1 = (3 \times 10^8 \text{ m}) \left( 6.58 \times 10^{-25} \text{ GeV} \right) = 1.97 \times 10^{-16} \text{ m} \cdot \text{GeV}
\]  
(1.17)

\[
\Rightarrow 1 \text{ GeV}^{-1} = 1.97 \times 10^{-16} \text{ m}
\]  
(1.18)

Summarizing the dimensionality:

\[
\text{time} = \text{length} = \frac{1}{\text{energy}}
\]  
(1.19)
1.2.2 Barns

We will later see later that when calculating cross sections, the conventional unit of area in particle physics is a barn.

\[ 1 \text{ barn} \equiv (10 \text{ fm})^2 = 10^{-24} \text{ cm}^2 \]  
\[ (1.20) \]

\[ 1 \text{ mb} \equiv 10^{-3} \text{ barns} = 10^{-27} \text{ cm}^2 \]  
\[ (1.21) \]

From (1.18), it can be shown that

\[ 1 \text{ GeV}^{-2} = 0.389 \text{ mb} \]  
\[ (1.22) \]

1.2.3 Electromagnetism

Finally, from \( \varepsilon_0 = 1 \) and \( c = 1 \)

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad \mu_0 = 1 \]  
\[ (1.23) \]

giving Maxwell’s equations the following form.

Field Tensor:

\[ F^\mu^\nu \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \]  
\[ (1.24) \]

Homogeneous:

\[ \partial_\mu F^\nu_\lambda + \partial_\nu F^\lambda_\mu + \partial_\lambda F^\mu_\nu = 0 \]  
\[ (1.25) \]

Inhomogeneous:

\[ \partial_\mu F^\mu^\nu = J^\mu \]  
\[ (1.26) \]

1.3 Relativistic Kinematics

1.3.1 Lorentz Invariant Phase Space

1.3.2 Mandelstam Variables
Chapter 2

Variation of Fields

2.1 The Field Worldview

We will see that the dynamical variable in quantum field theory is the field itself. A field is a mathematical concept that has a number or a construction of numbers (vector, spinor, tensor) defined at every point in spacetime. We assume that the fields are smoothly varying. To simplify notation, in these first few sections, we will denote a general field by \( \phi_a(x) \), where \( a \) indexes all of the components of \( \phi \), be they components of vectors, spinors, etc. or some direct product of them.

2.2 Variation

By studying the variation of fields we will discover the effect the Principle of Least Action has on the dynamics (the Euler-Lagrange equation) and we will discover the effects the symmetries of spacetime have on the dynamics (Noether’s Theorem). But first, we need to derive some properties of variation of functions in general.

We define the following.

\[
\delta x^\mu \equiv x'^\mu - x^\mu
\]

\[
\delta_\sigma \phi(x) \equiv \phi'(x) - \phi(x)
\]

\[
\delta \phi(x) \equiv \phi'(x') - \phi(x)
\]

\( \delta x^\mu \) is the difference in the values of coordinates referring to the same spacetime point but in different coordinate frames. \( \delta_\sigma \phi(x) \) is the value of the field
\(\phi(x)\) subtracted from the value of a field with a slightly different functional form, \(\phi'(x)\), evaluated at the same spacetime point, in the same coordinates. \(\delta\phi(x)\) is the difference in the functions \(\phi'(x')\) and \(\phi(x)\) evaluated in in different coordinate systems.

Using the above definitions we can derive the following.

\[
\delta\phi(x) = \phi'(x + \delta x) - \phi(x) \\
= \phi'(x) + \delta x^\mu \partial_\mu \phi'(x) + \ldots - \phi(x) \\
\simeq \delta_o \phi(x) + \delta x^\mu \partial_\mu \phi'(x) \\
= \delta_o \phi(x) + \delta x^\mu \partial_\mu (\phi(x) + \delta_o \phi(x)) \\
= \delta_o \phi(x) + \delta x^\mu \partial_\mu \phi(x) + \ldots
\]

We have used that the variations are small and that field is smooth by Taylor expanding \(\phi'(x)\). Therefore, to leading order in the variation, we have

\[
\delta\phi(x) = \delta_o \phi(x) + \delta x^\mu \partial_\mu \phi(x) \tag{2.1}
\]

Similarly for a function of several variables, we have

\[
\delta_o f(x, y) \equiv f'(x, y) - f(x, y) \\
\delta f(x, y) \equiv f'(x', y') - f(x, y)
\]

\[
\delta f(x, y) = f'(x + \delta x, y + \delta y) - f(x, y) \\
= f'(x, y) + \delta x^\mu \frac{\partial}{\partial x^\mu} f'(x, y) + \delta y^\mu \frac{\partial}{\partial y^\mu} f'(x, y) + \ldots - f(x, y) \\
\simeq \delta_o f(x, y) + \delta x^\mu \frac{\partial}{\partial x^\mu} f'(x, y) + \delta y^\mu \frac{\partial}{\partial y^\mu} f'(x, y) \\
= \delta_o f(x, y) + \delta x^\mu \frac{\partial}{\partial x^\mu} (f(x, y) + \delta_o f(x, y)) + \delta y^\mu \frac{\partial}{\partial y^\mu} (f(x, y) + \delta_o f(x, y)) \\
= \delta_o f(x, y) + \delta x^\mu \frac{\partial}{\partial x^\mu} f(x, y) + \delta y^\mu \frac{\partial}{\partial y^\mu} f(x, y) + \ldots
\]

\[
\therefore \delta f(x, y) = \delta_o f(x, y) + \delta x^\mu \frac{\partial}{\partial x^\mu} f(x, y) + \delta y^\mu \frac{\partial}{\partial y^\mu} f(x, y) \tag{2.2}
\]
One can begin to see that variation, $\delta$, follows similar operational rules as that of the differential operator, $d$. Indeed, we can derive a product rule for variation as follows.

Let $f(x, y) \equiv g(x) h(y)$

$$\Rightarrow \quad \delta f(x, y) = \delta_o(g(x) h(y)) + h(y) \delta x^\mu \frac{\partial}{\partial x^\mu} g(x) + g(x) \delta y^\mu \frac{\partial}{\partial y^\mu} h(y)$$

$$\delta_o(g(x) h(y)) = g'(x) h'(y) - g(x) h(y)$$

$$= (g(x) + \delta_o g(x))(h(y) + \delta_o h(y)) - g(x) h(y)$$

$$= g(x) h(y) + (\delta_o g(x)) h(y) + g(x) \delta_o h(y) + (\delta_o g(x))(\delta_o h(y)) - g(x) h(y)$$

$$= \frac{\partial^2}{\partial x^\mu \partial y^\mu} O[\delta^2]$$

$$\Rightarrow \quad \delta f(x, y) = h(y) \left( \delta_o g(x) + \delta x^\mu \frac{\partial}{\partial x^\mu} g(x) \right) + g(x) \left( \delta_o h(y) + \delta y^\mu \frac{\partial}{\partial y^\mu} h(y) \right)$$

$$\therefore \quad \delta (g(x) h(y)) = h(y) \delta g(x) + g(x) \delta h(y) \quad (2.3)$$

One can show that $\delta_o$ commutes with partial derivatives as follows.

$$\partial_\mu \delta_o \phi(x) = \partial_\mu (\phi'(x) - \phi(x))$$

$$= \partial_\mu \phi'(x) - \partial_\mu \phi(x)$$

$$= \delta_o \partial_\mu \phi(x)$$

$$\therefore \quad \delta_o \partial_\mu \phi(x) = \partial_\mu \delta_o \phi(x) \quad (2.4)$$

### 2.3 The Principle of Least Action

The Classical Mechanics of a field can be described by introducing a Lagrangian density (often just called a Lagrangian), $L$, a function of the field and its first derivatives$^1$.

$$L(x) = L(\phi_\alpha(x), \partial_\mu \phi_\alpha(x)) \quad (2.5)$$

$^1$It can be shown that $L$ cannot depend on higher derivatives of the field if the theory is to remain causally consistent.
The Principle of Least Action from Classical Mechanics states that the dynamics of a system obeying the physics described by some Lagrangian is such that the action functional is minimized. The **action** functional is defined by

$$ S[\phi_a, \partial_\mu \phi_a] \equiv \int d^4x \, \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) $$

(2.6)

Let us consider variation by perturbing the field, but leaving the coordinates alone.

$$ \delta_o \phi_a(x) = \phi'_a(x) - \phi_a(x) $$

Using the properties of variation that we have derived, we have

$$ \delta_o S = \int d^4x \, \delta_o \mathcal{L} $$

$$ = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta_o \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta_o (\partial_\mu \phi_a) \right] $$

$$ = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta_o \phi_a - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta_o \phi_a + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta_o \phi_a \right) \right] $$

$$ = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta_o \phi_a - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \right] \delta_o \phi_a + \int d\sigma_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta_o \phi_a $$

Note that terms with repeated $a$ have an implied sum over $a$. The surface integral above is zero because $\delta_o \phi_a = 0$ on the boundary. The action is minimized at a critical point.

$$ \delta_o S = 0 $$

Minimizing the action for arbitrary $\delta_o \phi_a$ gives the **Euler-Lagrange Equation**.

$$ 0 = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \right] \delta_o \phi_a $$

$$ \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) = 0 $$

(2.7)

Given a Lagrangian, the Euler-Lagrange Equation can be used to derive the equation of motion for the field.
2.4 Noether’s Theorem

Now let us allow for variations where the coordinates transform infinitesimally, called a **diffeomorphism**.

\[ \delta x^\mu = x'^\mu - x^\mu \]

Then the Lagrangian varies like any other function of the coordinates.

\[
\delta L = \delta_o L + \delta x^\mu \partial_\mu L = \partial_\mu \left( \frac{\delta L}{\partial \phi_a} \delta_o \phi_a + \frac{\partial L}{\partial (\partial_\mu \phi_a)} \delta_o (\partial_\mu \phi_a) + \delta x^\mu \partial_\mu L \right) 
\]

The above term is zero because \( \phi_a(x) \) that is the physical solution satisfies the Euler-Lagrange Equation. Therefore, the change in a Lagrangian under a general diffeomorphism is given by the following.

\[
\delta L = \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi_a)} \delta_o \phi_a \right) + \delta x^\mu \partial_\mu L \tag{2.8} 
\]

Let us consider only those diffeomorphisms that are **symmetries** of physics. That is, those transformations that leave the equations of motion invariant. This is guaranteed only if the diffeomorphism leaves the action invariant. Note that because a diffeomorphism is a change in coordinates, in general, the volume element, \( d^4x \), also varies.

\[
\delta S = \int \left( \delta (d^4x) L + d^4x \delta L \right) \tag{2.9} 
\]

The change in the volume element is given by the following.

\[
\delta (d^4x) = (\partial_\mu \delta x^\mu) d^4x \tag{2.10} 
\]
Therefore

\[
\delta S = \int d^4x \left[ (\partial_\mu \delta x^\mu) \mathcal{L} + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right) + \delta x^\mu \partial_\mu \mathcal{L} \right]
\]

\[
= \int d^4x \partial_\mu \left[ \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right]
\]

using equation (2.1), \( \delta \phi_a = \delta \phi_a - \delta x^\mu \partial_\mu \phi_a \)

\[
= \int d^4x \partial_\mu \left[ \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (\delta \phi_a - \delta x^\mu \partial_\mu \phi_a) \right]
\]

\[
= \int d^4x \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a + \left( \mathcal{L} \eta^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\nu \phi_a \right) \delta x_\nu \right]
\]

Let us define the energy-momentum tensor, whose convenience will become apparent, as

\[
T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\nu \phi_a - \mathcal{L} \eta^{\mu\nu}
\] (2.11)

Then

\[
\delta S = \int d^4x \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - T^{\mu\nu} \delta x_\nu \right]
\]

Now define the Noether current as

\[
\mathcal{J}^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - T^{\mu\nu} \delta x_\nu
\] (2.12)

Because physical symmetries leave the action invariant for any region of spacetime, we can derive a conservation law as follows.

\[
\delta S = \int d^4x \partial_\mu \mathcal{J}^\mu = 0
\]

\[
\Rightarrow \partial_\mu \mathcal{J}^\mu = 0
\] (2.13)

Therefore, for each diffeomorphism that is a symmetry of physics, there exist a conserved Noether current.

Expanding the Einstein sum in the conservation law gives

\[
\partial_\mu \mathcal{J}^\mu = \partial_0 \mathcal{J}^0 - \vec{\nabla} \cdot \vec{\mathcal{J}} = 0
\]
We define the Noether charge as

\[ Q \equiv \int d^3 x \mathcal{J}^0 \]  

(2.14)

Consider the time derivative of the Noether charge. Using Stokes’ Theorem, this can be converted into a surface integral. For large enough regions, on physical grounds, we expect the flux of the spacial Noether current to be zero across the boundary.

\[ \frac{dQ}{dt} = \int d^3 x \partial_0 \mathcal{J}^0 = \int d^3 x \nabla \cdot \mathbf{J} = \oint d\mathbf{\sigma} \cdot \mathbf{J} = 0 \]

Therefore the Noether charges are conserved in time.

\[ \frac{dQ}{dt} = 0 \]  

(2.15)

They are a formal expression for the Classically conserved quantities in physics like energy and momentum. Noether’s theorem links each of these conserved quantities to a physical symmetry.

### 2.5 Spacetime Translation

Consider a diffeomorphism that does not mix components of the fields. All it does is shift the coordinates each by a constant, \( c^\mu \).

\[ x'^\mu = x^\mu + c^\mu \]  

(2.16)

\[ \Rightarrow \delta x^\mu = c^\mu \]  

(2.17)

After the transformation, the value of the field is the same for the same spacetime point.

\[ \delta \phi_a = 0 \]  

(2.18)

Then, the definition of the Noether current, equation (2.12), gives

\[ \mathcal{J}^\mu = -T^{\mu\nu} \delta x_\nu = -T^{\mu\nu} c_\nu \]  

(2.19)
Because translating spacetime coordinates by an arbitrary constant is an observed symmetry in Nature, Noether’s theorem says that the corresponding Noether currents are conserved.

\[ \partial_\mu J^\mu = 0 \]

The arbitrariness of \( c_\nu \) implies

\[ \partial_\mu T^{\mu\nu} = 0 \quad (2.20) \]

We define the **total four-momentum** of the field \( \phi_a \) as the corresponding Noether charges.

\[ P^\nu \equiv \int d^3x \; T^{0\nu} \quad (2.21) \]

The **four-momentum density**, \( \mathcal{P}^\nu \), is defined by \( T^{0\nu} \), such that

\[ P^\nu \equiv \int d^3x \; \mathcal{P}^\nu \quad (2.22) \]

Using the definition of the energy-momentum tensor, equation (2.11), we have

\[ \mathcal{P}^\nu = \frac{\partial L}{\partial (\partial_0 \phi_a)} \partial^\nu \phi_a - L \; \eta^{0\nu} \quad (2.23) \]

We define the **conjugate momentum** of a field as follows.

\[ \pi_a \equiv \frac{\partial L}{\partial \dot{\phi}_a} \quad (2.24) \]

Then, in this notation, we have

\[ \mathcal{P}^\nu = \pi_a \partial^\nu \phi_a - L \; \eta^{0\nu} \quad (2.25) \]

The temporal component of the four-momentum density is the **Hamiltonian density**, denoted \( \mathcal{H} \equiv \mathcal{P}^0 \). The temporal component of the four-momentum is the **Hamiltonian**, representing the total energy of the field.

\[ H = P^0 = \int d^3x \left( \pi_a \dot{\phi}_a - L \right) = \int d^3x \; \mathcal{H} \quad (2.26) \]
The remaining spacial components are the three components of the total momentum of the field.

\[ \vec{P} = -\int d^3x \pi_a \vec{\nabla} \phi_a \] (2.27)

Energy and momentum conservation are a consequence of the translation symmetry of spacetime.

### 2.6 Lorentz Transformations

The Lorentz transformations have a unitary representation, where $\Lambda^\mu_\nu$ are the components of a real antisymmetric matrix.

\[ x'^\mu = L^\mu_\nu x^\nu = e^{\Lambda^\mu_\nu} x^\nu \] (2.28)

Therefore, for infinitesimal $\Lambda^\mu_\nu$, the transformation can be written as

\[ x'^\mu \simeq (\delta^\mu_\nu + \Lambda^\mu_\nu) x^\nu \] (2.29)

\[ \Rightarrow \quad \delta x^\mu = \Lambda^\mu_\nu x^\nu \] (2.30)

There exists a set of spin matrices, $\left(\Sigma^{\alpha\beta}\right)_{ab}$, labeled by four-vector indices, $\alpha$ and $\beta$, but the indices of a single matrix are $a$ and $b$. The spin matrix relates the transformation of the spacetime four-vector coordinates to the transformation of the components of the field under a Lorentz transformation.

\[ \phi'_a(x') = \left(\delta^\mu_\nu + \frac{1}{2} \left(\Sigma^{\alpha\beta}\right)_{ab} \Lambda_{\alpha\beta}\right) \phi_b(x) \] (2.31)

Note that there is implied summation over all paired indices: $b$, $\alpha$, and $\beta$.

\[ \Rightarrow \quad \delta \phi_a(x) = \frac{1}{2} \left(\Sigma^{\alpha\beta}\right)_{ab} \Lambda_{\alpha\beta} \phi_b(x) \] (2.32)

Plugging in equations (2.30) and (2.32) into equation (2.12) for the Noether current gives

\[ J^\mu = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \left(\Sigma^{\alpha\beta}\right)_{ab} \Lambda_{\alpha\beta} \phi_b - T^\mu_\alpha \Lambda_{\alpha\beta} x^\beta \]
Using the antisymmetry of $\Lambda_{\alpha\beta}$, we have

$$J^\mu = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \left( \Sigma^\alpha_{\alpha\beta} \right)_{\alpha\beta} \Lambda_{\alpha\beta} \phi_b - \frac{1}{2} \left( T^\mu_{\alpha\beta} x^\beta - T^\mu_{\mu\beta} x^\alpha \right) \Lambda_{\alpha\beta}$$

Now factoring out the arbitrary infinitesimal $\Lambda_{\alpha\beta}$ and the factor of $\frac{1}{2}$ gives

$$J^\mu = \frac{1}{2} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \left( \Sigma^\alpha_{\alpha\beta} \right)_{\alpha\beta} \phi_b - \left( T^\mu_{\alpha\beta} x^\beta - T^\mu_{\mu\beta} x^\alpha \right) \right] \Lambda_{\alpha\beta}$$

Let

$$M^{\mu\alpha\beta} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \left( \Sigma^\alpha_{\alpha\beta} \right)_{\alpha\beta} \phi_b - \left( T^\mu_{\alpha\beta} x^\beta - T^\mu_{\mu\beta} x^\alpha \right)$$

Then

$$J^\mu = \frac{1}{2} M^{\mu\alpha\beta} \Lambda_{\alpha\beta}$$

Because the Lorentz transformations are an observed symmetry in Nature, Noether’s theorem says that $\partial_\mu J^\mu = 0$. The arbitrariness of $\Lambda_{\alpha\beta}$ means that there is a conserved current for each $\alpha$ and $\beta$.

$$\partial_\mu M^{\mu\alpha\beta} = 0$$

The corresponding Noether charges are then given by

$$M^{\alpha\beta} \equiv \int d^3 x M^{\alpha\beta}$$

Plugging in equations (2.25), (2.24), and (2.33) gives

$$M^{\alpha\beta} = \int d^3 x \left[ \pi_a \left( \Sigma^\alpha_{\alpha\beta} \right)_{\alpha\beta} \phi_b + \left( x^\alpha \mathcal{P}^\beta - x^\beta \mathcal{P}^\alpha \right) \right]$$

Like all Noether charges, each of these are conserved in time. The asymmetry of $\Lambda_{\alpha\beta}$ implies that both $\Sigma^{\alpha\beta}$ and $M^{\alpha\beta}$ are asymmetric as well.

Upon multiplication by $\frac{1}{2}$ and a Levi-Civita symbol, the spacial terms with the four-momentum density would give a cross product, $\vec{x} \times \vec{\mathcal{P}}$. This motivates us to interpret them as the orbital angular momentum. With this in mind, notice that the $\Sigma$ term mixes the components of the field, a transformation in a space internal to the field as opposed to spacetime. This leads one to recognize this term as a component of angular momentum that
is internal to the field, which will later be interpreted as spin of particles when the field is quantized.

\[ M^{\alpha\beta} = \int d^3x \left[ \pi_a (\Sigma^{\alpha\beta})_{ab} \phi_b + \left( x^\alpha P^\beta - x^\beta P^\alpha \right) \right] \] (2.38)

The angular momentum is given by

\[ J^k = \frac{1}{2} \epsilon^{ijk} M_{ij} \] (2.39)

Angular momentum conservation is a consequence of the rotation symmetry of spacetime, part of Lorentz invariance. The \( M^{0i} \) charges, derived from the transformations mixing space and time components, correspond to Lorentz boosts. They too are conserved, but are not as physically apparent as angular momentum.

### 2.7 Internal Symmetries

In addition to the symmetries of spacetime, some fields have symmetries relating their internal degrees of freedom. These types of transformations have the following unitary representation

\[ \phi'_a(x) = e^{i(G_r)_{ab} \theta_r} \phi_b(x) \] (2.40)

where \( G_r \) are hermitian matrices, such that the transformation is unitary. (Note that summation is implied over \( r, a, \) and \( b \).) \( G_r \) are called the “generators” of the transformation, and \( \theta_r \) are the corresponding parameters. Note that this type of transformation has nothing to do with spacetime. Instead, it directly mixes the components of the field. If this type of transformation leaves the equations of motion invariant, and is therefore a symmetry, then physics is unaffected by whether phenomena are described by \( \phi_a(x) \), or a field related to \( \phi_a(x) \) by such a transformation. Fields related to one another by such a transformation are called different “gauges” of the same field, and this type of symmetry is known as “gauge invariance.”

For infinitesimal \( \theta_r \), the transformation can be expanded as

\[ \phi'_a(x) \simeq (\delta_{ab} + i (G_r)_{ab} \theta_r) \phi_b(x) \] (2.41)
\[ \Rightarrow \quad \delta \phi_a(x) = i \ (G_{r})_{ab} \ \theta_r \ \phi_b(x) \quad (2.42) \]

Plugging this into equation (2.12) gives the Noether currents.

\[ J^\mu_r = i \ \frac{\partial L}{\partial (\partial_\mu \phi_a)} \ (G_r)_{ab} \ \theta_r \ \phi_b(x) \quad (2.43) \]

If the Lagrangian is gauge invariant, then those Noether currents are conserved, and the corresponding Noether charges (derived using equations (2.14) and (2.24)) are conserved in time.

\[ Q_r = \int d^3 x \ i \ \pi_a \ (G_r)_{ab} \ \theta_r \ \phi_b \quad (2.44) \]
Chapter 3

The Free Real Scalar Field

We will investigate the simplest example of a field, a real scalar field. That is, one that has a single component that is a real number and therefore transforms trivially under the Lorentz transformations. We will use this simple example to explain the canonical quantization procedure that can then be applied to any other type of field. After studying the Classical properties of the real scalar field, we will summarize the principles of quantum mechanics, and finally quantize the real scalar field.

3.1 Classical Theory

The Classical theory of a field has two inputs: the type of field, and the Lagrangian that describes its dynamics. The field we are now studying is the real scalar field that we will denote $\phi(x)$. The Lagrangian$^1$ describing the free relativistic dynamics of this field is

$$L = \frac{1}{2} \left( \partial_{\mu} \phi \left( \partial^{\mu} \phi \right) - m^2 \phi^2 \right)$$  \hspace{1cm} (3.1)

where $m$ is a real constant.

The first step in studying the Classical theory of a field is to calculate its equation of motion and its Noether charges. The equation of motion is given by the Euler-Lagrange equation, equation (2.7).

$$\left( \partial^2 + m^2 \right) \phi = 0$$  \hspace{1cm} (3.2)

$^1$The appropriateness of this Lagrangian will become apparent when we see that it insures that the field is relativistic by satisfying equation (1.11).
This is called the **Klein-Gordon equation**. The Hamiltonian and momentum are given by equations (2.26) and (2.27).

\[
H = \int d^3x \frac{1}{2} \left( \pi^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right)
\]  
(3.3)

\[
\vec{P} = -\int d^3x \pi \vec{\nabla} \phi
\]  
(3.4)

This implies that the four-momentum density is given by

\[
\mathcal{P}^0 = \mathcal{H} = \frac{1}{2} \left( \pi^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right)
\]  
(3.5)

and

\[
\mathcal{P}^\mu = -\pi \vec{\nabla} \phi
\]  
(3.6)

which is used in equation (2.37) to calculate \(M^{\mu\nu}\).

\[
M^{\mu\nu} = \int d^3x (x^\mu \mathcal{P}^\nu - x^\nu \mathcal{P}^\mu)
\]  
(3.7)

Note that the spin term in \(M^{\mu\nu}\) is zero because the real scalar field does not change under the Lorentz transformation and therefore equation (2.31) implies that the spin matrix, \(\Sigma\), is zero.

We next turn our attention studying the solutions of the Klein-Gordon equation. Consider the following.

\[
f_k(x) \equiv \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-i k \cdot x}
\]  
(3.8)

where

\[
k^\mu \equiv (\omega_k, \vec{k})^\mu
\]  
(3.9)

and

\[
\omega_k \equiv +\sqrt{k^2 + m^2}
\]  
(3.10)

Plugging in the **plane-waves** \(f_k(x)\) in for \(\phi(x)\) shows that they are solutions.

The functions \(f_k(x)\) form a complete basis for a complex function space. The appropriate inner product in this space involves the following operation.

\[
\vec{a} \partial_0 \vec{b} \equiv a \partial_0 b - (\partial_0 a) b
\]  
(3.11)
The completeness relation is derived by plugging equations (3.8) and (3.11) into the following.

\[
\int d^3x \, f^*_k(x) i \, \partial^0 \, f_k(x) = \frac{i}{(2\pi)^3 \, 2 \sqrt{\omega_k \, \omega_k'}} \int d^3x \, e^{i \, k' \cdot x} \partial^0 e^{-i \, k \cdot x} - \left( \partial^0 e^{i \, k' \cdot x} \right) e^{-i \, k \cdot x} \\
= \frac{\omega_k + \omega_k'}{2 \sqrt{\omega_k \, \omega_k'}} \frac{1}{(2\pi)^3} \int d^3x \, e^{i (k' - k) \cdot x} \\
= \frac{\omega_k + \omega_k'}{2 \sqrt{\omega_k \, \omega_k'}} e^{i (\omega_{k'} - \omega_k) t} \frac{1}{(2\pi)^3} \int d^3x \, e^{-i (k' - k) \cdot x} \\
= \frac{2 \, \omega_k}{2 \, \omega_k} e^0 \, \delta^3(\vec{k} - \vec{k}')
\]

\[
\therefore \int d^3x \, f^*_k(x) i \, \partial^0 \, f_k(x) = \delta^3(\vec{k} - \vec{k}')
\]

(3.12)

### 3.2 Principles of Quantum Mechanics

### 3.3 Quantum Theory